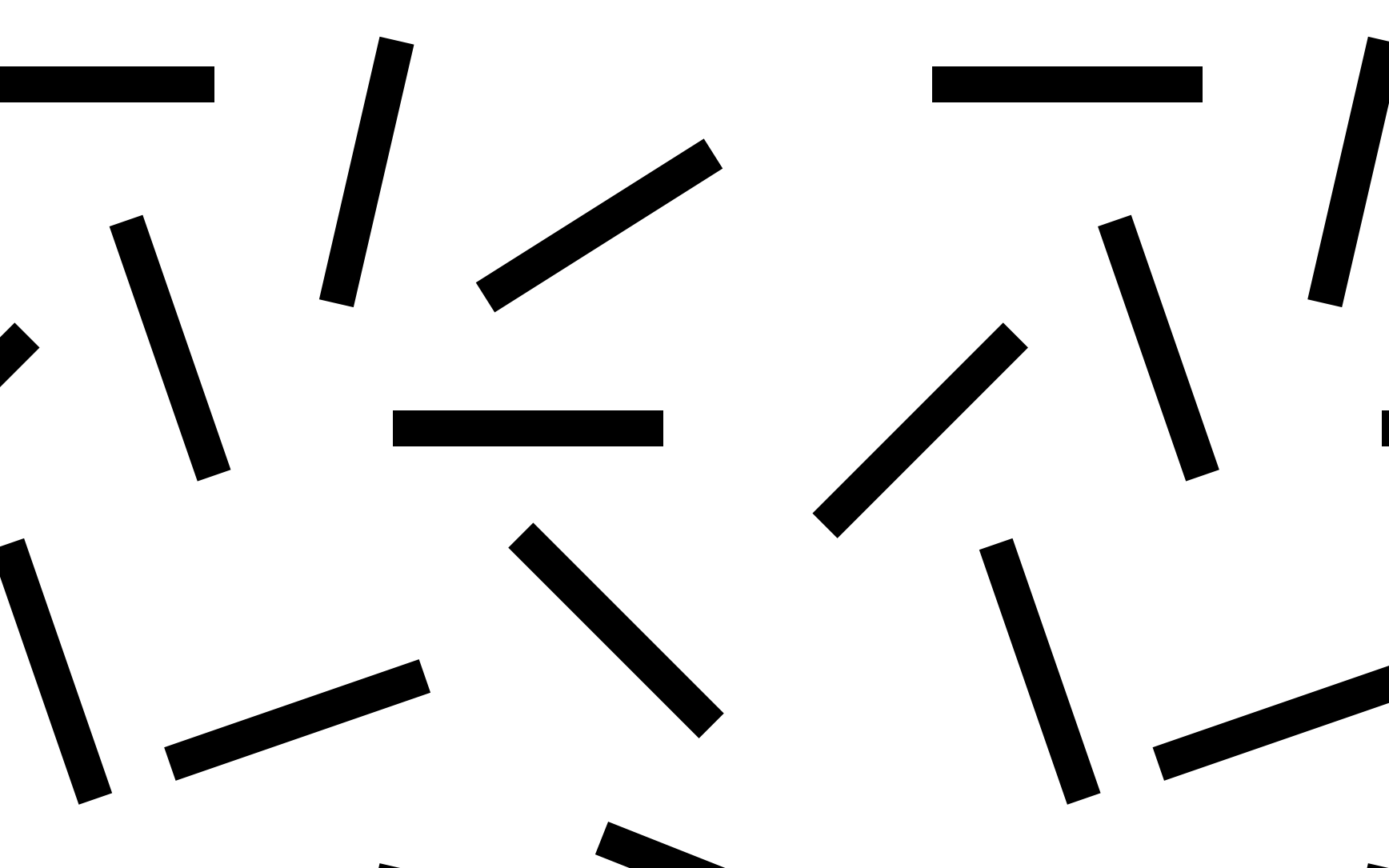


The 2 token theorem

Karoliina Lehtinen Aditya Prakash

2025

PROBLEM:

Given a parity automaton, is it history deterministic?

known results:

$n^{O(\log n)}$ time [Henzinger, Piterman '06]

NP HARD [Prakash '24]

PTIME for some subclasses

coBüchi automaton – Kuperberg, Skrzypczak '15

Büchi Automaton – Bagnol, Kuperberg '18

coBüchi Automaton – Boker, Kuperberg, Lehtinen, Skrzypczak '20

} 2 token games

proofs were not constructive

However, Bagnol, Kuperberg 2018 made a CONJECTURE:

2 TOKEN CONJECTURE

For every parity automaton A,

A is history-deterministic

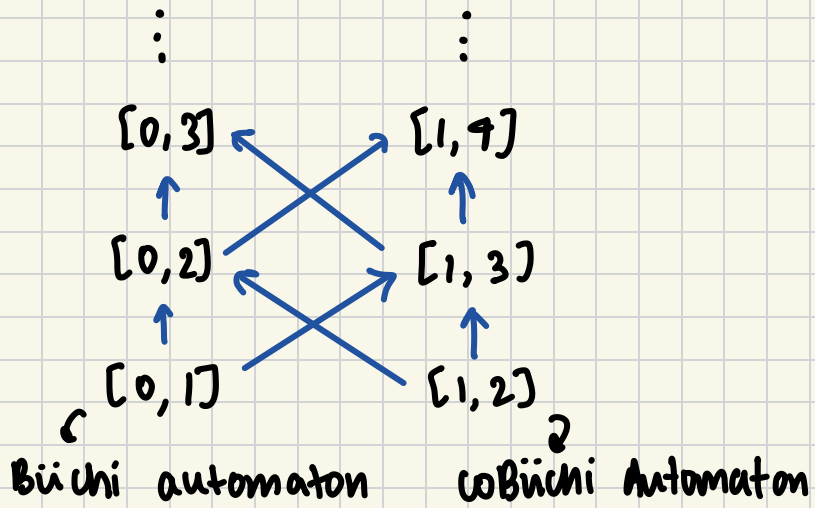
$\Leftrightarrow$

Eve wins the 2 token game on A

PARITY AUTOMATON

Our wining condition : a run is accepting iff the least priority occuring ∞ly often is even

[i,j] automaton: priorities in $\{i, i+1, \ldots, j\}$



PARITY - INDEX HIERARCHY

History Determinism:

an accepting run can be constructed on the fly, without knowing the full word

HD game: Played on an Automata A

2 players Adam ~ Eve

Adam plays letters

Eve Moves token

Eve's token starts at the start state



History Determinism: an accepting run can be constructed on the fly, without knowing the full word

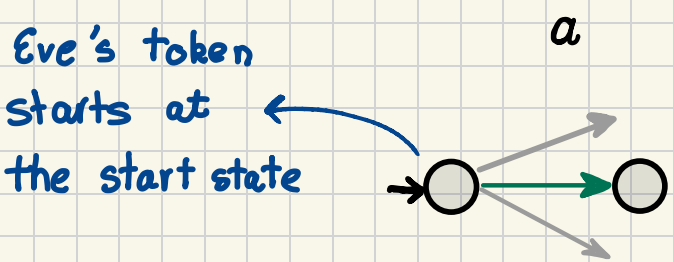
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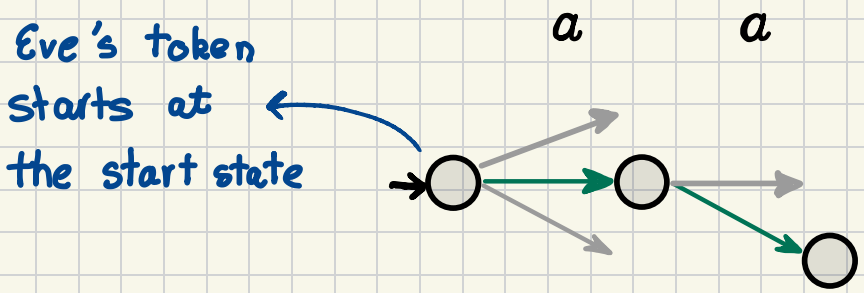
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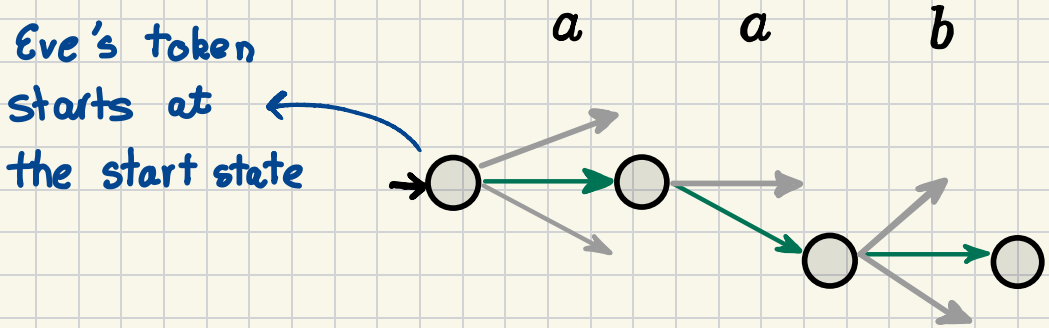
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2 players Adam & Eve

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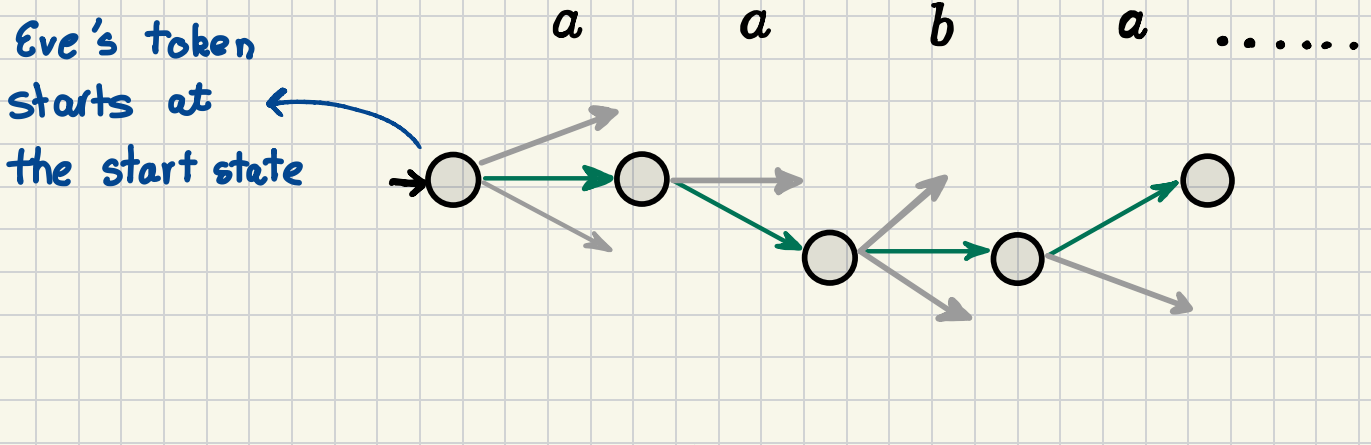
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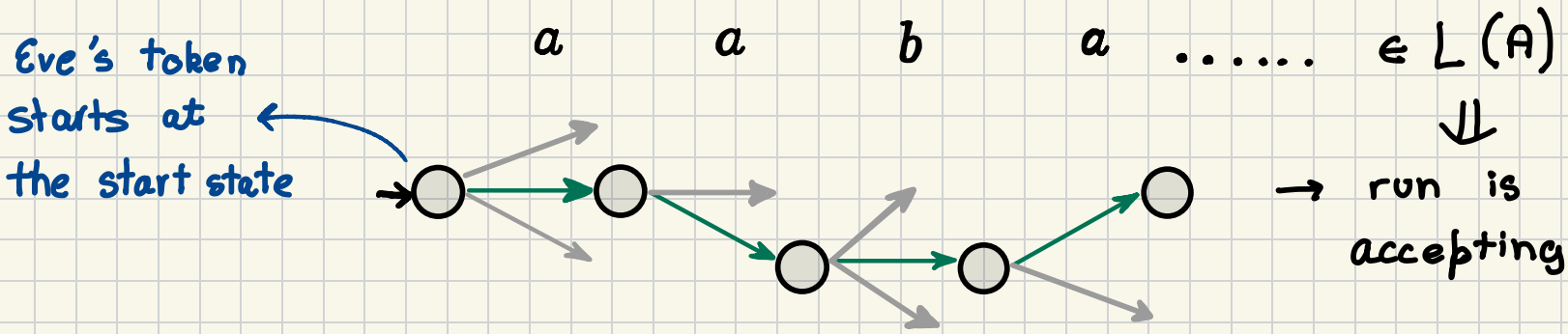
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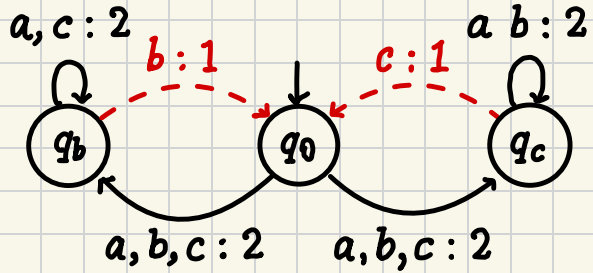
Adam plays letters

Eve Moves token



$\mathcal{A}$ is History Deterministic if Eve wins HD game

Examples

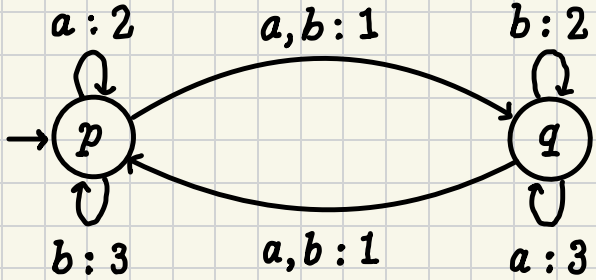


History Deterministic

$L = \{$ finite bs v finite c $\}$

Eve can alternate btw

$q_b \sim q_c$



Not History Deterministic

$L = \{a, b\}^{\omega}$

→ adam can stop Eve from taking any even edge

The 2 Token game

2-token game on Automata $\mathcal{A}$

The 2 Token game

2-token game on Automata $\mathcal{A}$

Starts with an Eve's token $\wedge$ 2 of Adam's tokens at $\rightarrow q_0$

Adam:

Eve : q_0

Adam : q_0

Adam : q_0

The 2 Token game

2-token game on Automata $\mathcal{A}$

Starts with an Eve's token $\wedge$ 2 of Adam's tokens at $\rightarrow q_0$

in each round

1. Adam selects a letter a_i
2. Eve moves her token $q_i \xrightarrow{a_i, c_i} q_{i+1}$

Adam: a

Eve : $q_0 \rightarrow q_1$

Adam: q_0

Adam: q_0

The 2 Token game

2-token game on Automata $\mathcal{A}$

Starts with an Eve's token $\wedge$ 2 of Adam's tokens at $\rightarrow (q_0)$

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2. Eve moves her token $q_i \xrightarrow{a_i, c_i} q_{i+1}$
3. Adam moves his 2 tokens $p_i^1 \rightarrow p_{i+1}^1$, $p_i^2 \rightarrow p_{i+1}^2$

Adam: a

Eve : $q_0 \rightarrow q_1$

Adam: $q_0 \rightarrow p_1^1$

Adam: $q_0 \rightarrow p_1^2$

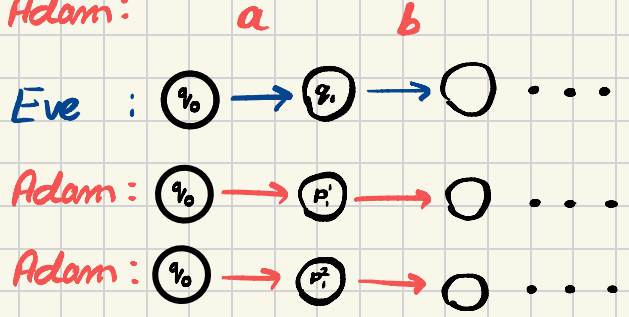
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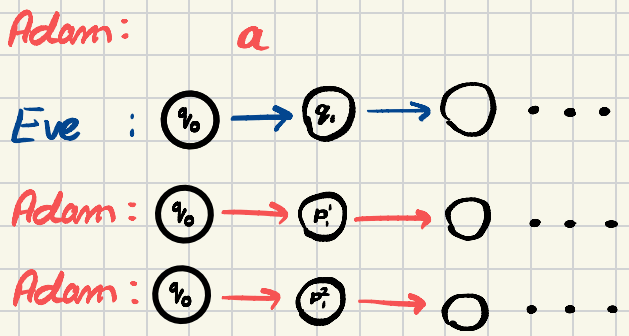
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in each round

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3. Adam moves his 2 tokens $p_i^1 \rightarrow p_{i+1}^1$, $p_i^2 \rightarrow p_{i+1}^2$



Eve wins: if either of Adam's runs are accepting

then Eve's run is accepting

General Notation

→ $G1((A, q); (B, p))$

↳ 1 token game where

Eve starts on state q of Automaton A

Adam starts on state p of Automaton B

→ $Gk((A, q); (B_1, p_1) \dots (B_k, p_k))$

HD $\Longrightarrow$ 2 token

if A is HD $\Longrightarrow$ Eve builds an accepting run on all accepting words

$\Downarrow$

by the same strategy

Eve wins the 2 token game

However, Bagnol, Kuperberg 2018 made a CONJECTURE:

2 TOKEN CONJECTURE

For every parity automaton A,

A is history-deterministic

$\Leftrightarrow$

Eve wins the 2 token game on A

Solving 2 Token Game for Parity Automaton

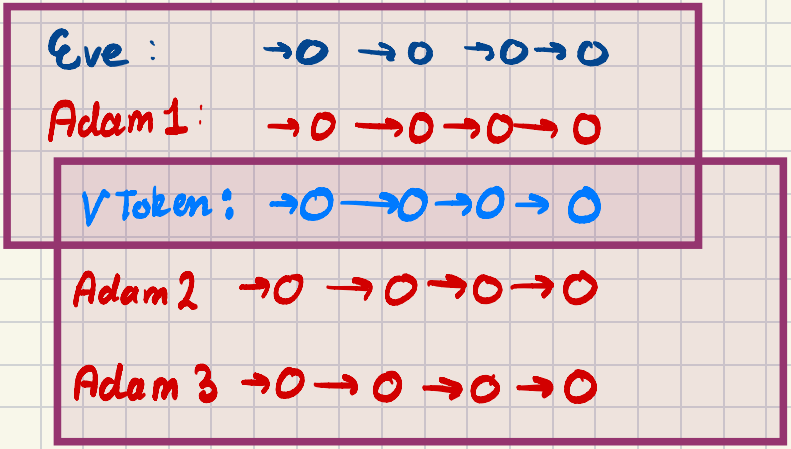
① Fixed Parity → Polynomial Time (Udi Boker, Denis Kupferman, Michał Skrzypczak 2020)

② Parity Index not fixed: PSPACE (Paul Hunter, Anuj Dawar 2005)

do we need more tokens?

short answer → NO!

Lemma (Bagnol, Kuperberg (2018)): Eve wins 2 tokens $\Leftrightarrow$ Eve wins k tokens, $k \geq 1$



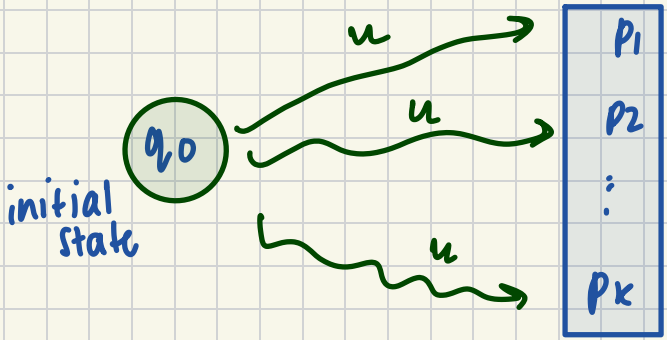
Eve win 2 token

$\Rightarrow$

Eve wins 3 token

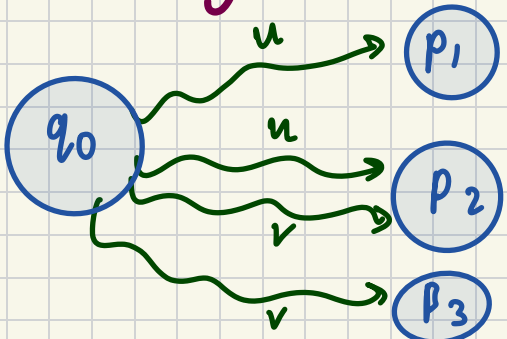
Coreachable $(p_1, p_2, \ldots, p_k)$

We say the tuple $(p_1, p_2, \ldots, p_k)$ is coreachable if $\exists\, u \in \Sigma^*$ such that



there are there are runs on u from q_0 to $p_1, p_2, \ldots, p_k$

Weakly Coreachable: transitive closure of co reachability



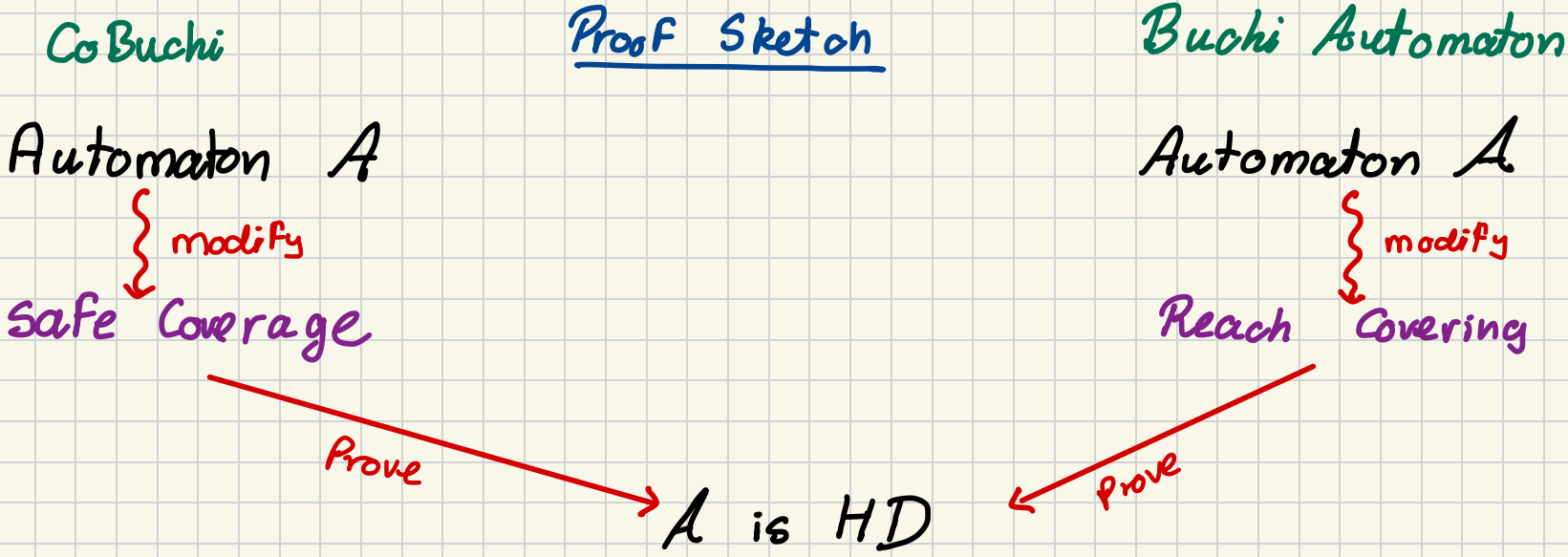
then we say $p_1 \& p_3$ are weakly coreachable.

(they need not be reachable from q_0 on the same word)

$CR^*(A, p)$ = equivalence class containing p.

Theorem: For every Buchi or CoBuchi automaton A if Eve wins the 1-Token game from all co-reachable states on A then A is HD

Proof Sketch



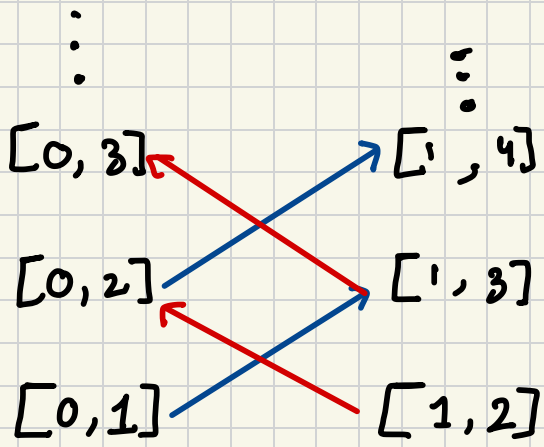
Induction Step

Even to odd

2 token ⇒ HD
For [0, k]

⇓

2 token ⇒ HD
for [1, k+2]



odd to Even

2 token ⇒ HD
for [1, k]

⇓

2 token ⇒ HD
for [0, k]

∴ 2 Token Theorem !!

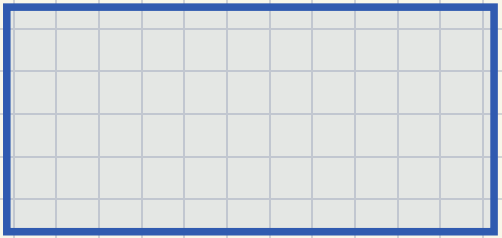
THEOREM 3.1: *Let $\mathcal{A}$ be a parity automaton on which Eve wins the two token game. Then, there is a* **simulation equivalent** *subautomaton $\mathcal{B}$ of $\mathcal{A}$ such that*

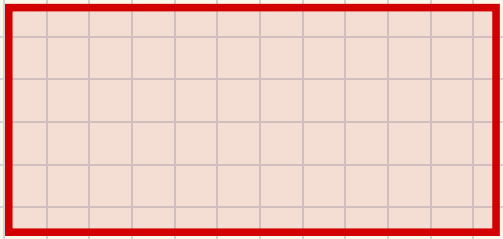
- *Eve wins $Gk(q; p_1, \dots p_k)$ in $\mathcal{B}$ $\forall$ $k+1$ states $q, p_1, \dots, p_k$ that are weakly coreachable in $\mathcal{B}$*
- *If $\mathcal{B}$ is history deterministic $\Rightarrow$ $\mathcal{A}$ is HD*

Simulation Equivalence

→ Formally defined using Simulation game

Intuition:



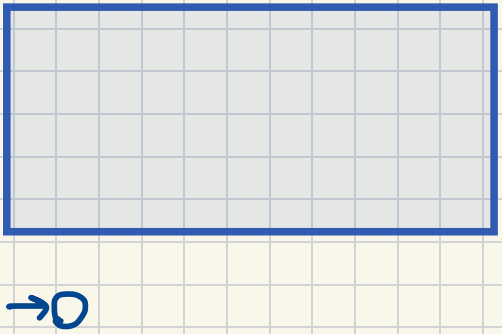


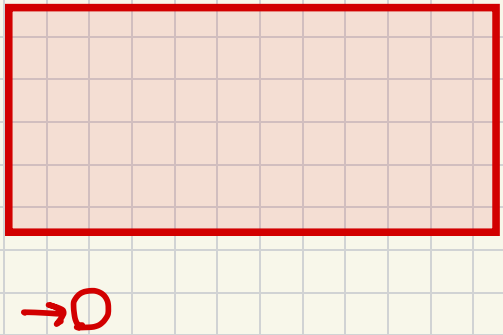
→D

Simulation Equivalence

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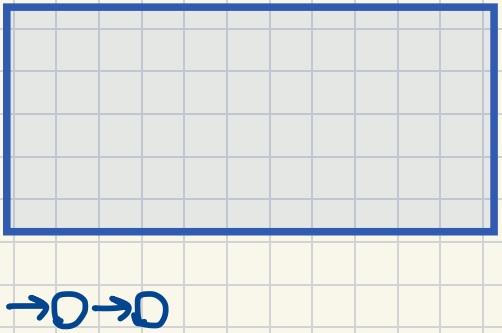


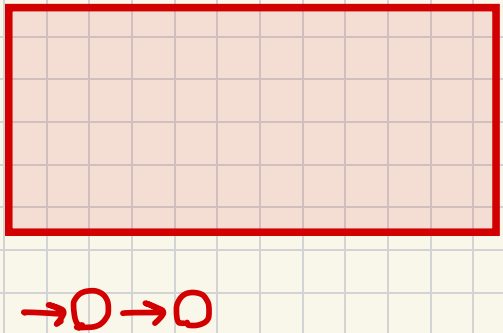


Simulation Equivalence

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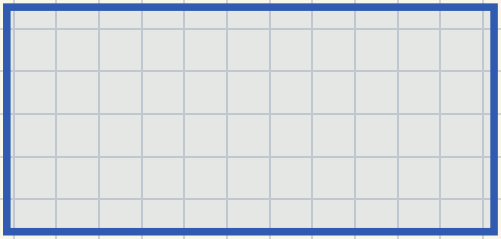


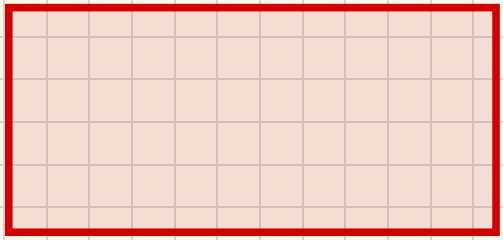


Simulation Equivalence

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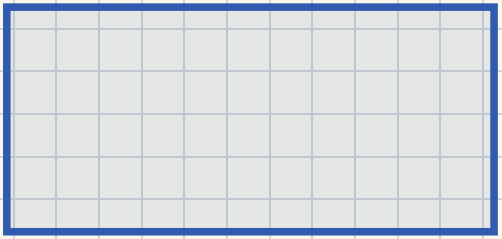
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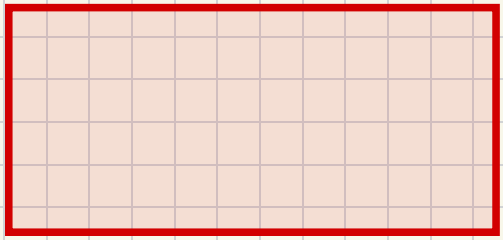
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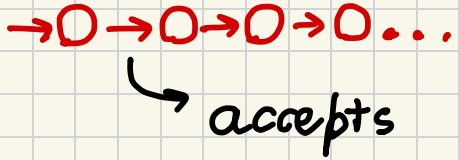
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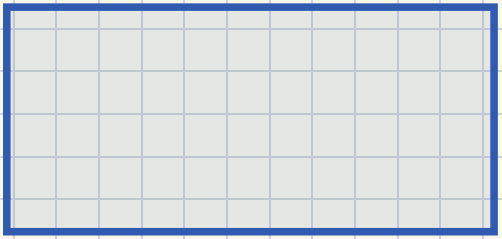


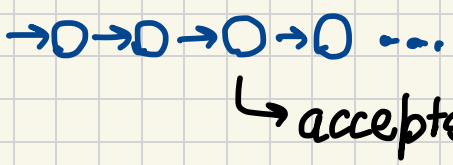


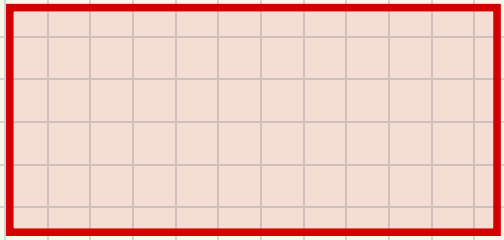
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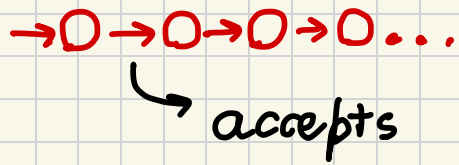
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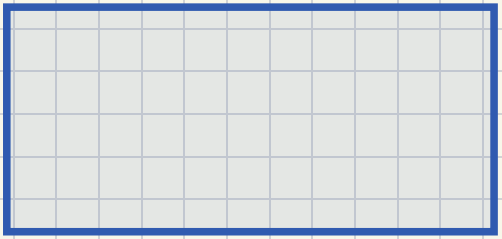




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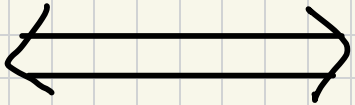
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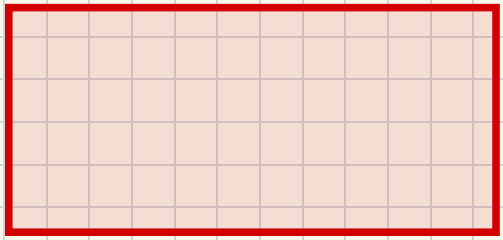


→○→○→○→○ ...

↳ accepts



equivalence



→○→○→○→○...

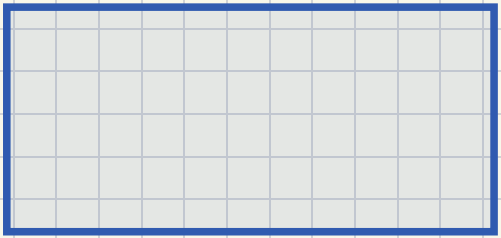
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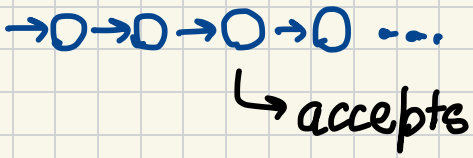
A is HD ⟺ B is HD

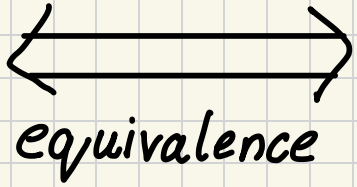
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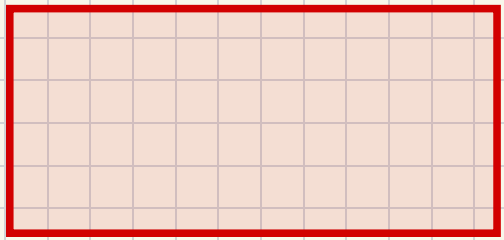
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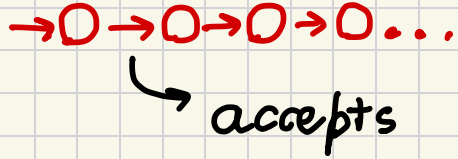
Intuition:











THEOREM 3.1: *A* parity automaton on which Eve wins the two token game. Then, there is a simulation equivalent subautomaton $\mathcal{B}$ of $\mathcal{A}$ such that

- Eve wins $G_k(q_i; p_1, \dots p_k)$ in $\mathcal{B}$ $\forall$ $k+1$ states $q, p_1, \dots, p_k$ that are weakly coreachable in $\mathcal{B}$

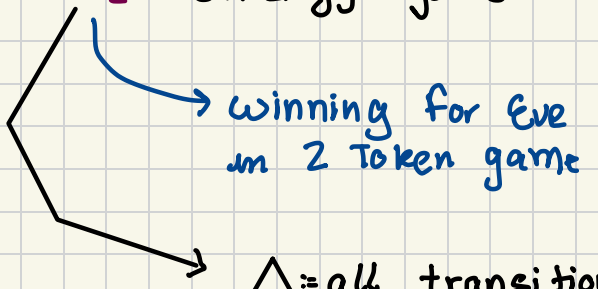
- ✓ If $\mathcal{B}$ is history deterministic $\Rightarrow$ $\mathcal{A}$ is HD

→ we will construct such $\mathcal{B}$

Eve wins 2 Token game on $\mathcal{A}$

$\Rightarrow$ Eve wins 3 token game by strategy σ_3

$\therefore$ σ_2: strategy gained from σ_3 by assuming Adam's 3rd token copies Eve's token



$\mathcal{B}$:= sub automata of $\mathcal{A}$ only consisting of Δ transitions

NOTE: $\mathcal{B} \longleftrightarrow \mathcal{A}$

simulation equivalent

Eve wins $G_k(q; p_1, \dots p_k)$ in B $\forall$ $k+1$ states $q, p_1, \dots, p_k$ that are weakly coreachable in B

① coreachable $\Rightarrow$ weakly coreachable

transitivity of 1-token games

if Eve wins $G_k(\mathcal{A}; B_1 .., B_i, ..., B_k) \wedge G_1(B_i, B)$

$\Rightarrow$ Eve wins $G_k(\mathcal{A}; B_1 .., B, ..., B_k)$

Eve wins $Gk(q; p_1, \dots p_k)$ in B $\forall$ k+1 states $q, p_1, \dots, p_k$ that are ~~weakly coreachable~~ coreachable in B

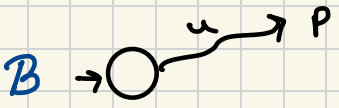
Proof by Induction

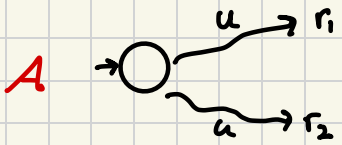
we will prove for 2 tokens

Eve wins $G2(q_i ; p_1, p_2)$ in B $\forall$ states q, p_1, p_2 that are ~~weakly coreachable~~ coreachable in B

Proof by Induction

$\forall u \in \Sigma^*$ Eve wins $G3((B, p) ; (A, r_1), (A, r_2), (A, p))$

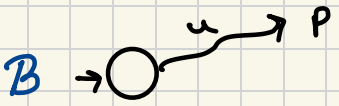




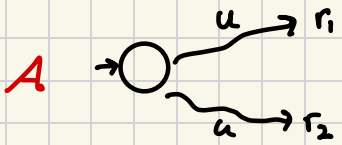
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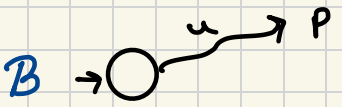
BASE $\longrightarrow$ $u = \varepsilon$



Eve wins $G2(q_i ; p_1, p_2)$ in B $\forall$ states q, p_1, p_2 that are ~~weakly coreachable~~ coreachable in B

Proof by Induction

$\forall u \in \Sigma^*$ Eve wins $G3((B, p) ; (A, r_1), (A, r_2), (A, p))$



Induction Step → $u_{k+1} = u_k \cdot a$

Eve wins $G3((B, pre(p)), (A, pre(r_1)), (A, pre(r_2)), (A, pre(p))$

$\Rightarrow G3((B, p'), (A, r_1), (A, r_2), (A, p))$

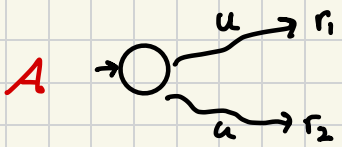
but since $pre(p) \rightarrow p$ is in B

$\exists\ q_1\ q_2$ s.t

$\Rightarrow G3((B, pre(p)), (A, q_1), (A, q_2), (A, pre(p))$

$\Rightarrow G1((B, p) ; (A, p'))$

$\Rightarrow G3((B, p), (A, r_1), (A, r_2), (A, p))$ □



Co-Buchi Automaton

&

Even to odd Induction Step

Priority Normalisation Procedure

For a co-Büchi Automaton A

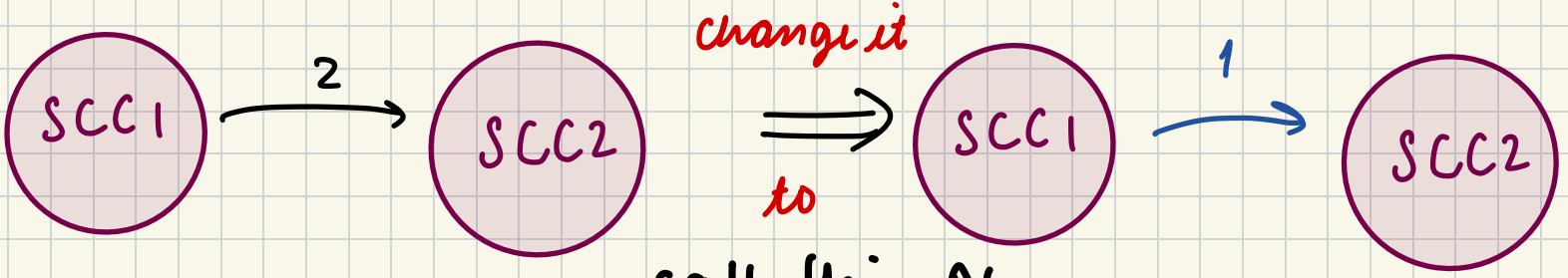
↓

we first change it to the equivalent [1,2] parity automaton

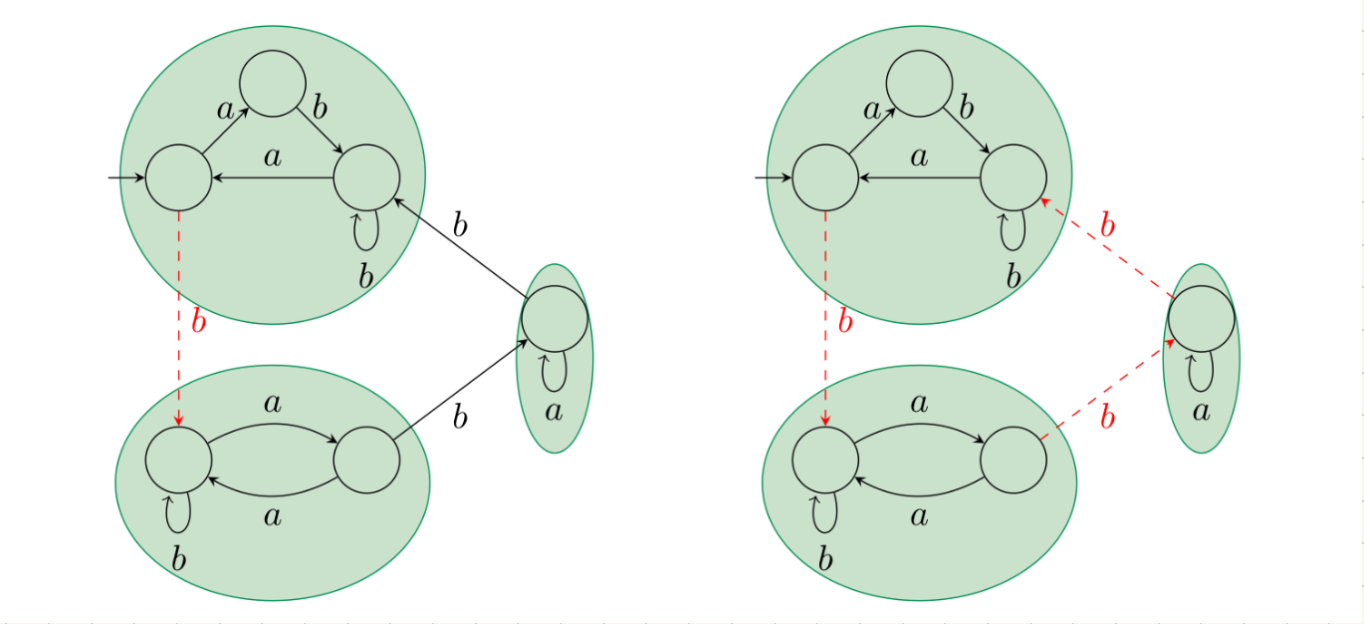
We make a Graph $G_{\to 1}$ from A → same set of vertices

→ add only priority 2 edges

Now for a priority 2 transition that is from one SCC to another

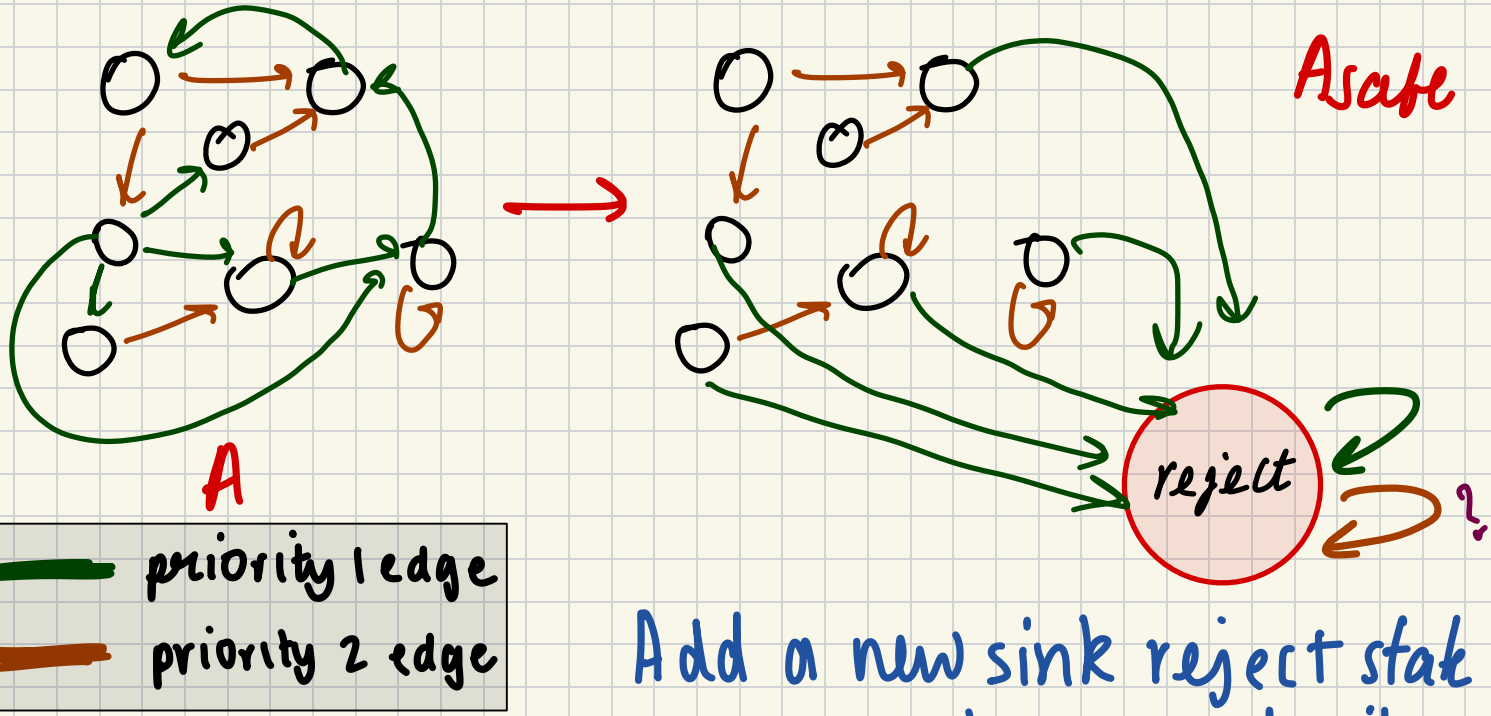


call this A'



CLAIM: A run is accepting in A iff a run in A' is

A_{safe} : safety automaton



Add a new sink reject state
redirect all priority 1 edges to it

Safe Coverage: We say A has safe coverage if

$\forall$ states p $\exists$ state $q \in CR^*(A, p)$

(p and q are weakly coreachable)

such that Eve wins $G_1(q, p)$ in A_{safe}.

Adam plays from p

Eve plays from q

LEMMA: Every co-Büchi automaton on which Eve wins $Gl(s,t)$ $\forall s,t$ where s,t are coreachable and is priority normalised $\Rightarrow$ it has safe coverage

Proof.

suppose it - is priority normalised
- does NOT have safe coverage



we will show

Eve does NOT win $Gl(s,t)$ (coreachable)

NO safe coverage $\Rightarrow$ $\exists$ state q st $\forall p \in CR^*(A, q)$

σ ← (winning strategy) Adam wins $G1(q;q)$ in A_{safe} $\Leftarrow$ **Adam wins $G1(p;q)$ in A_{safe}**

We will show a winning strategy for Adam for $G1(q, q)$ in A

(Note that this will show a contradiction since q is trivially weakly coreachable with itself)

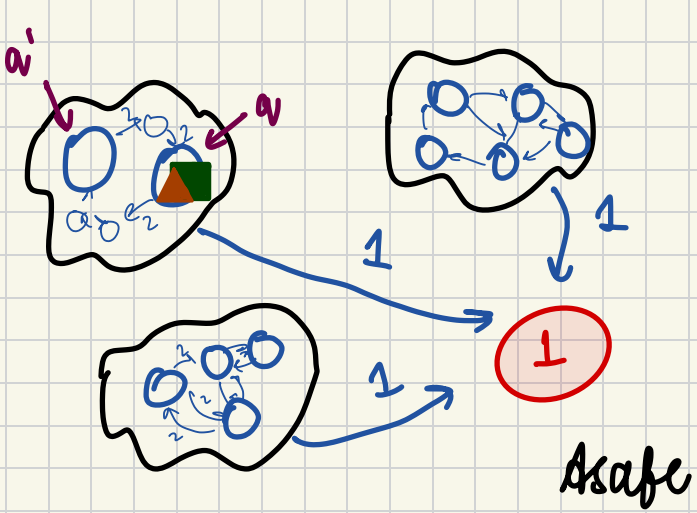
$G1(q;q)$ in A ⟶ chooses transitions & letters according to σ

(it can do that)

Note:- What does it mean for Eve to loose in A_{safe}?

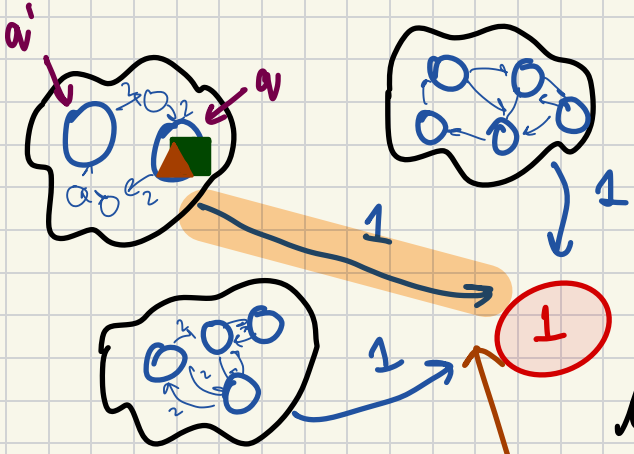
She will eventually have to take a 1 transition & go to the reject state and Adam did not go to the reject state. This implies Adam has a way to force Eve to eventually take a priority 1 edge while only taking priority 2 edges.

$\therefore$ σ ensures Eve will take a priority 1 edge eventually and Adam's token has reached some state q' only using priority 2 edges

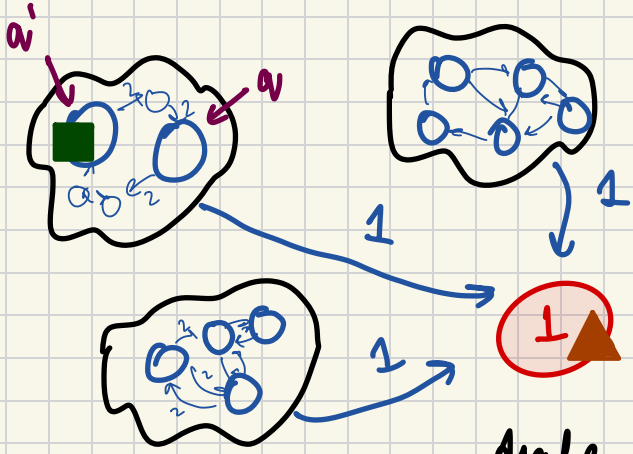


∴ σ ensures Eve will take a priority 1 edge eventually and Adam's token has reached some state q' only using priority 2 edges

Game start ..... Eventually .....>



Safe



Safe

Adam's token

Eve's token

Eve takes this transition

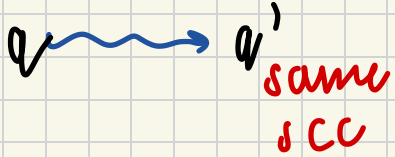
Adam has not changed his SCC and reached some q'

But what happens in A?

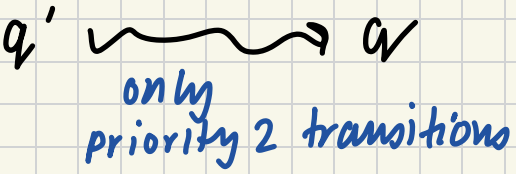
Recall that A is priority normalised

Adam has not changed his SCC in A

and reached some q'



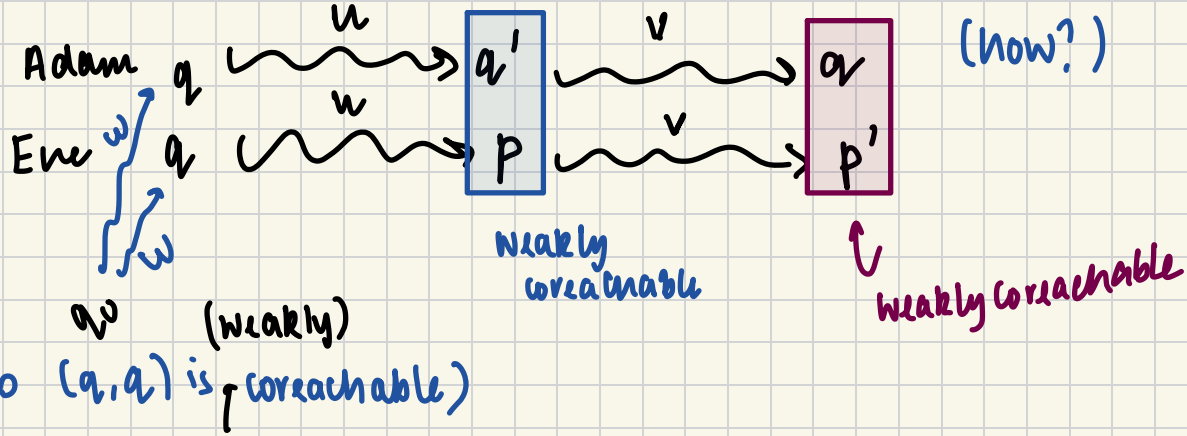
$\therefore$ there is a path back from q' to q



Now Adam's strategy in A: pick letters and transitions so that he can go back to q!

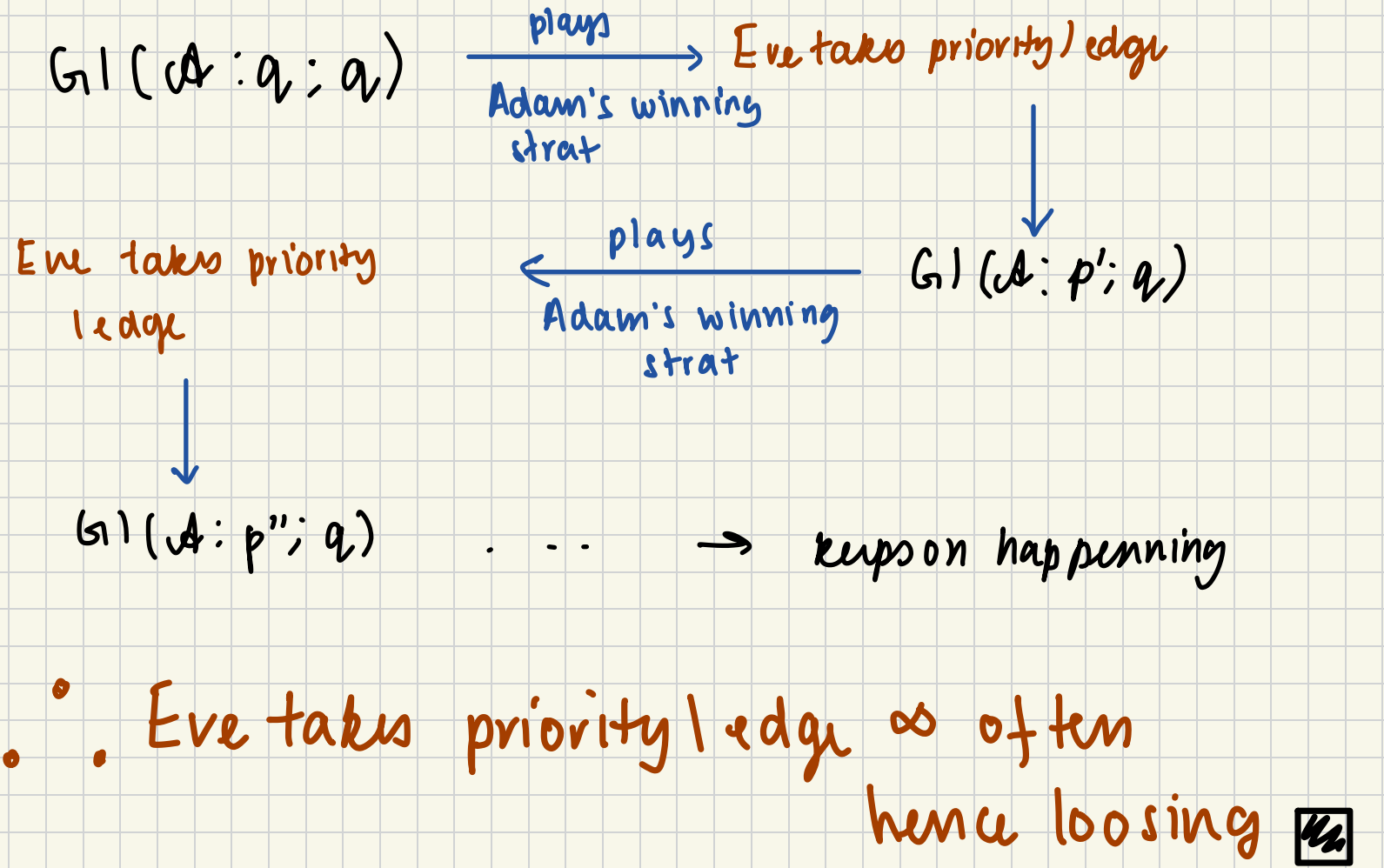
After Adam goes to q where is Eve ???

Eve is at some state p' weakly coreachable to q in A



By ASSUMPTION — (NO safe coverage) ⇒ Adam wins from $G_1(p', q)$ in A_{safe}

and now plays his winning strat for $G_1(p', q)$


Gl(A : q ; q)
plays
Adam's winning strat
Eve takes priority / edge
Gl(A : p' ; q)
plays
Adam's winning strat
Eve takes priority / edge
Gl(A : p'' ; q)
. . .
response happenning
∴ Eve takes priority / edge ∞ often
hence loosing

LEMMA: If Eve wins $Gl(A;B)$ for some parity automaton $A, B \longrightarrow$ then Eve has a uniform strategy

Prakash, 2025

LEMMA: Let A be a safety / reachability automaton. Following are equivalent :-

[BKS17, BL23a]

1) Eve wins the 1 token game on A
2) A history deterministic
3) Eve has a positional winning strategy in the HD game on A
4) A has language-equivalent deterministic subautomaton D

Theorem 3.2 A (coBuchi Automata)

Eve wins 1 token game $\Rightarrow$ A is HD

now we know by **Safe coverage**

$\forall p \; \exists q_1$ (weakly coreachable) . Eve wins $G1(p, q_1)$ in A_{safe}

$\Rightarrow \exists q_2$

$\Rightarrow \exists q_3$

- Eve wins $G1(q_1, q_2)$
- Eve wins $G1(q_2, q_3)$

$p \to q_1 \to q_2 \to q_3$

Theorem 3.2 A (coBuchi Automata)

Eve wins 1 token game $\Rightarrow$ A is HD

now we know by **Safe coverage**

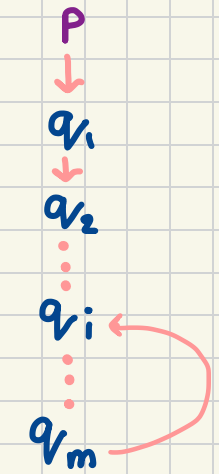
$\forall p \; \exists q_1$ (weakly coreachable) . Eve wins $G1(q_1, p)$ in A_{safe}

$\Rightarrow \exists q_2$. Eve wins $G1(q_2, q_1)$

$\Rightarrow \exists q_3$. Eve wins $G1(q_3, q_2)$

$\vdots$

$\Rightarrow \exists q_m$. Eve wins $G1(q_i, q_m)$



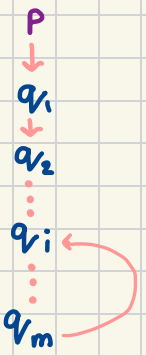
$\forall p \; \exists q_1$ (weakly coreachable) . Eve wins $G1(q_1, p)$ in A_{safe}

$\Rightarrow \exists q_2$. Eve wins $G1(q_2, q_1)$

$\Rightarrow \exists q_3$. Eve wins $G1(q_3, q_2)$

$\vdots$

$\Rightarrow \exists q_m$. Eve wins $G1(q_i, q_m)$



$\Rightarrow$ Eve wins $G1(q_i, q_i)$

Transitivity of G1

$\therefore \forall p \; \exists q$ s.t Even wins $G1(q, p) \wedge G1(q, q)$ in A_{safe}

$\Rightarrow (A_{safe}, q)$ can be determinised

q → safe deterministic states

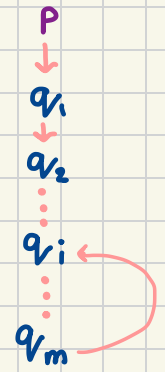
$\forall p \; \exists q_1$ (weakly coreachable) . Eve wins $G1(p, q_1)$ in A_{safe}

$\Rightarrow \exists q_2$. Eve wins $G1(q_1, q_2)$

$\Rightarrow \exists q_3$. Eve wins $G1(q_2, q_3)$

$\vdots$

$\Rightarrow \exists q_m$. Eve wins $G1(q_m, q_i)$



$\therefore \forall p \; \exists q$ s.t Even wins $G1(p, q) \wedge G1(q, q)$ in A_{safe}

$\Rightarrow (A_{safe}, q)$ can be determinised

q → safe deterministic states

$\therefore \forall p \; \exists q$ s.t Even wins $G1(p,q) \wedge G1(q,q)$ in A_{safe}

$\Rightarrow (A_{safe}, q)$ can be determinised

q → safe deterministic states

σ_{safe} : Positional HD strategy in A_{safe} $\forall$ safe deterministic states

$\therefore$ we can take all safe deterministic states

$\wedge$ transitions in σ_{safe}

to get D_{safe} deterministic safety automata

$$L(D_{safe}, q) = L(A_{safe}, q)$$

now we make a DAG D_w (w is a word)

D_w : states are of the form (u, q)

u is a prefix of w

q is active on u

$q \in CR^*(A, u)$ and is safe deterministic

there always $\exists$ such a state

now we make a DAG D_w (w is a word)

D_w : states are of the form (u, q)

u is a prefix of w

q is active on u

$(u, q) \longrightarrow (ua, q') \iff q \xrightarrow{a} q'$ in D_{safe}

q' not a rejecting sink state

say $w \in L(A)$

$\Rightarrow w = uw'$ → has an ∞ path in D_w

↳ only has 2-edges

HD Strategy for A

① σ_{safe} : Positional HD strategy in A_{safe}
∀ safe deterministic states

↳ $D_{safe} \longrightarrow D_\omega$

② σ_{G1} : Positional winning strategy for Eve
for 1 token game on A

∀ pairs of weakely coreachable states

HD Strategy for A

σ_{G1} $D_{safe} \longrightarrow D_\omega$

① Eve starts at initial state of A

↳ tracks r_0 st weak coreachable and safe in D_ω

adam plays a letter

→ Eve plays according to σ_{G1} → winning for 2 token

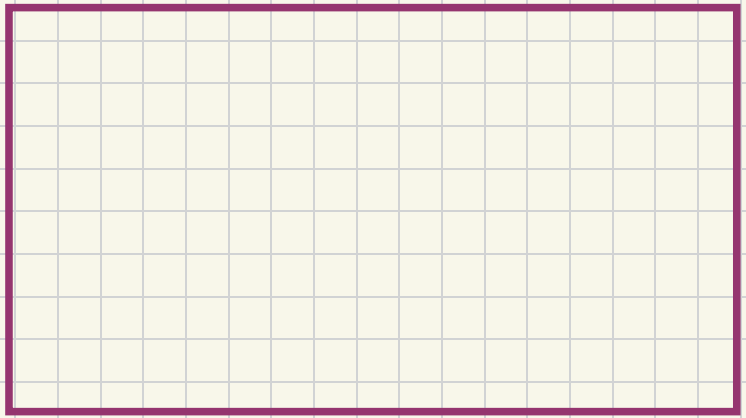
$r_0 \to r_1$ s.t $(\varepsilon, r_0) \to (a, r_1)$ in D_ω

Eve

$q_0 \xrightarrow{\sigma_{G_1}} q_1$

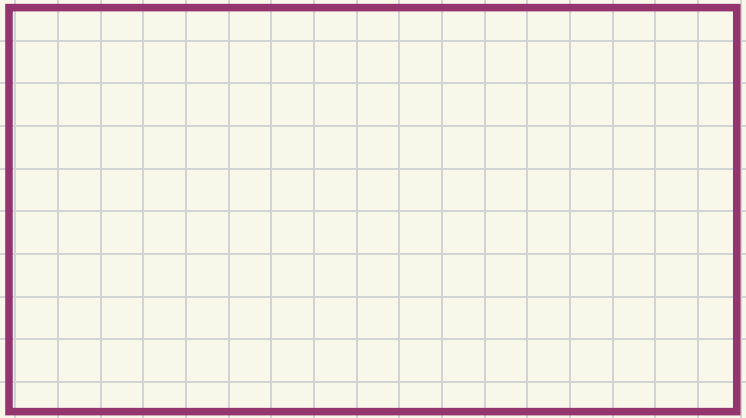
Virtual adam token follows D_ω

$r_0 \to r_1$



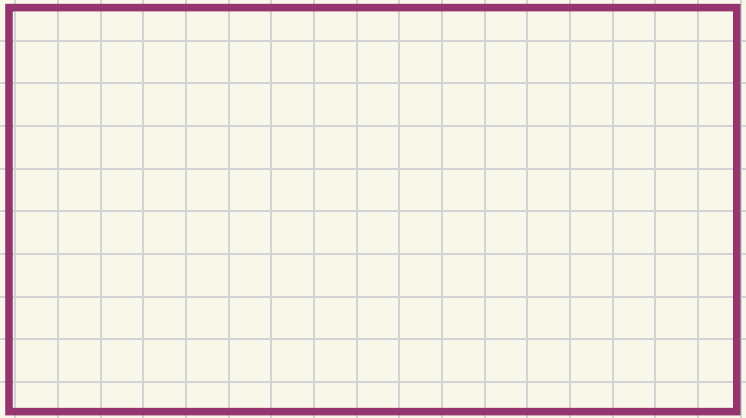
Adar :

(r_0, ε)



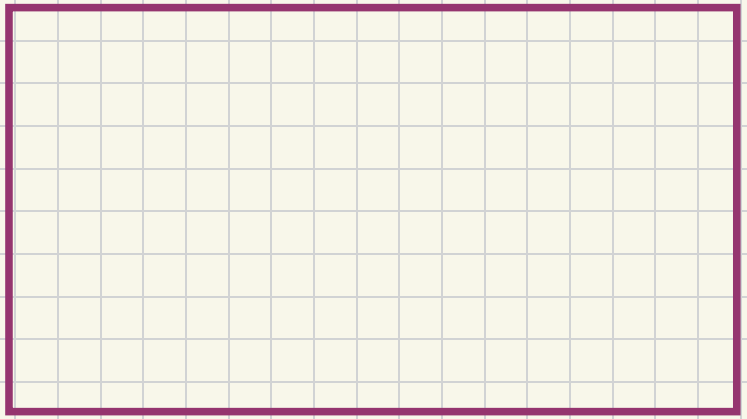
Adar : a

(r_0, ε)



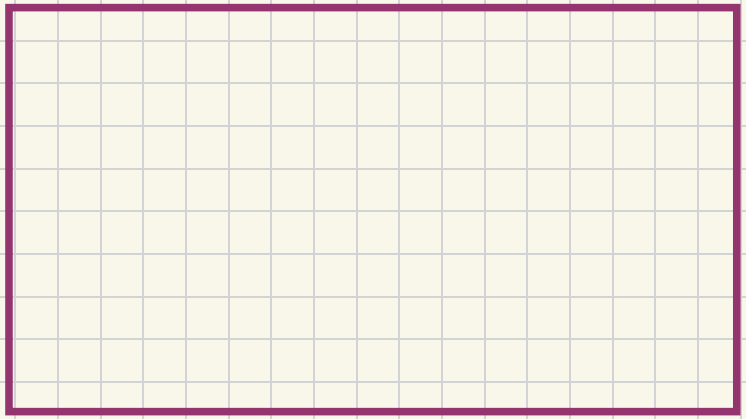
Adar : a

(r_0, ε)



Adan : a

$(r_0, \varepsilon) \longrightarrow (r_1, a)$

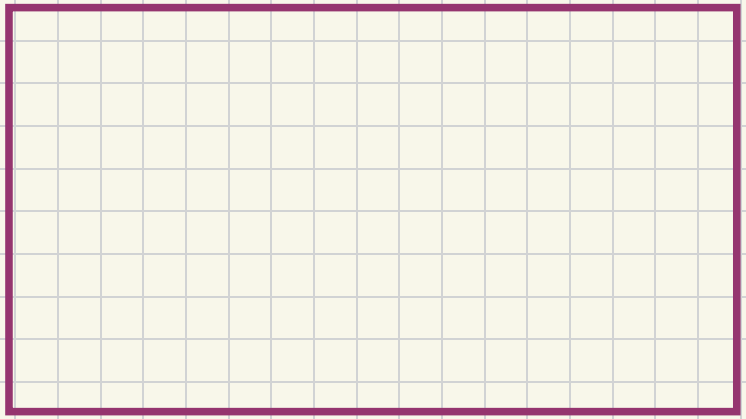


Adan : a b

$(r_0, \varepsilon) \longrightarrow (r_1, a) \longrightarrow (r_2, ab)$

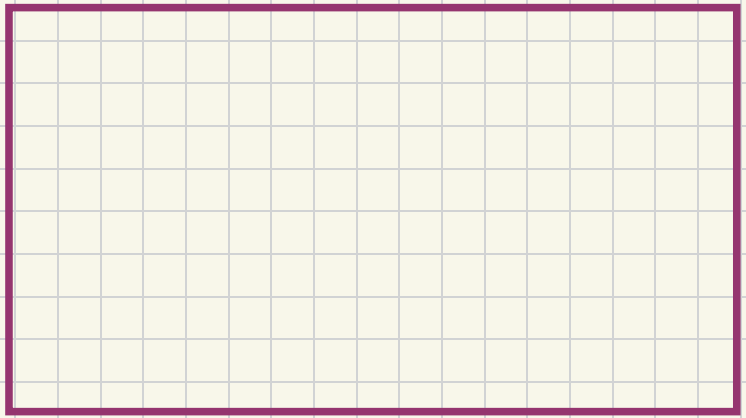
$(r_1', a) \longrightarrow (r_2', ab)$

active



Adan : u

$(r_0, \varepsilon) \longrightarrow \ldots \longrightarrow (r_n, u)$



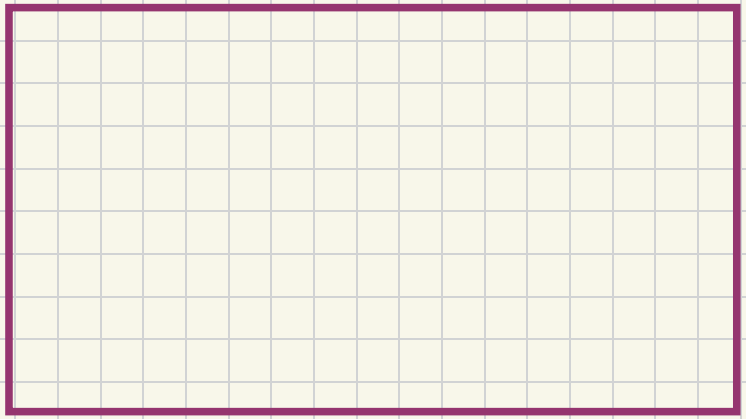
Adam : $u\,a$

$(r_0, \varepsilon) \longrightarrow \cdots \longrightarrow (r_n, u) \not\longrightarrow$

$\cdots (r_n', u)$

active

Longest such path



Adam: $\omega^\omega \in L(A)$

$\omega = uv$

↳ only sees 2-edges

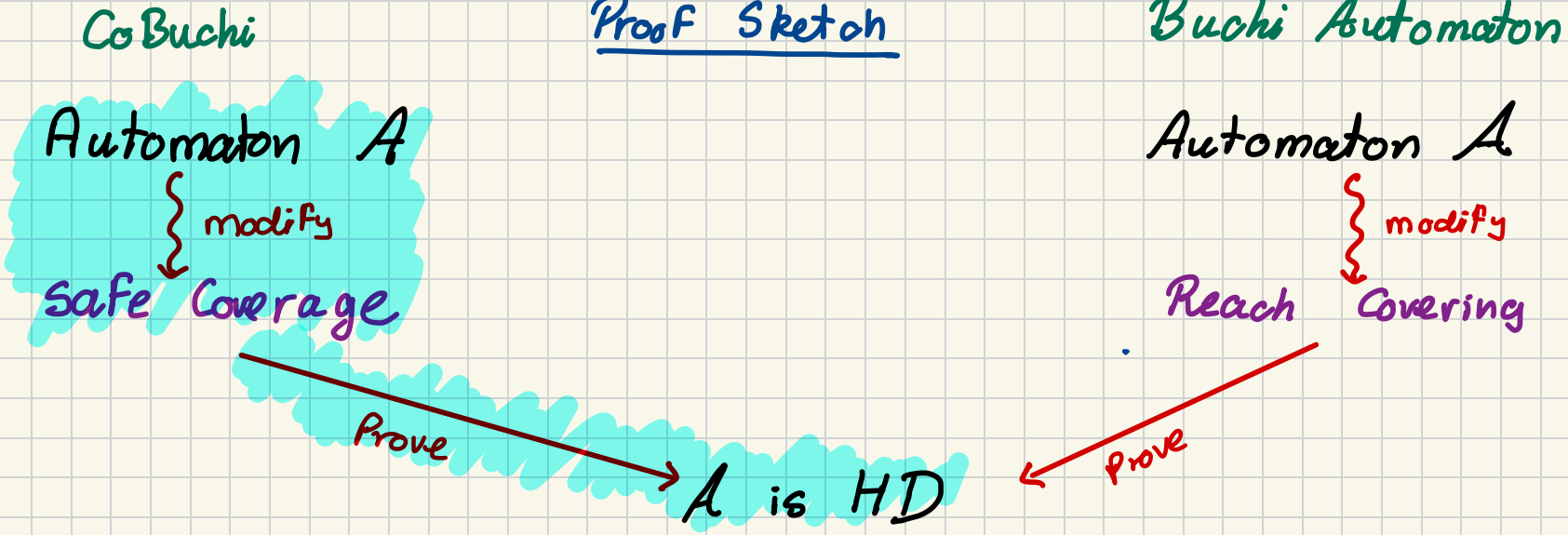
$(r_u, u) \to (r_{uq}, ua) \to \cdots \to \infty$

↳ virtual token will end up at an ∞ path

Eve wins in $G1(q, r_u)$ $\Rightarrow$ we have an HD winning strategy

Theorem: For every Buchi or CoBuchi automaton $\mathcal{A}$ if Eve wins the 1-Token game from all co-reachable states on $\mathcal{A}$ then $\mathcal{A}$ is HD

Proof Sketch



EVEN TO ODD INDUCTION STEP:

We show the following result -

THEOREM 3.5: Let $K \geq 1$ natural number st for every $[0,K]$ automaton $\mathcal{A}$, Eve wins the 2 token game on $\mathcal{A}$ iff A is HD

Then, for every $[1,K+2]$ automaton $\mathcal{A}$,
Eve wins 2 token game on $\mathcal{A}$ iff A is HD

win on $[0,K] \Rightarrow$ win on $[2,K+2]$

↓ we will show

$[1,K+2]$

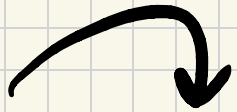
REQUIRED STEPS

COBÜCHI

Priority Normalization

$\mathcal{A}$ safe

safe coverage



$[1, K+2]$

2 priority reduction

$\mathcal{A}_{\neg}$: 2 approximation of $\mathcal{A}$

1 safe coverage

LEMMA 3.6' Every $[1, K+2]$ automaton $\mathcal{A}$ that is

1) 2-priority reduced
2) Eve wins 1token game from $\forall (s,t)$ coreachable

$\Downarrow$

Eve has safe-coverage

LEMMA 3.7' Every $[1, K+2]$ automaton $\mathcal{A}$ that is

1) 2-priority reduced
2) Eve wins 1token game from $\forall (s,t)$ coreachable

$\Downarrow$

Eve has 1 safe-coverage

WANT TO SHOW:

- 1 safe coverage $\Leftrightarrow$ Eve wins 2-token game from all (i.e.) coreachable

$\Downarrow$

HD

Naive approach (as we did for coBüchi) fails !!!

Why? A run in $\mathcal{A}_{\neg\varphi}$ that does not end up at the rejecting sink state need not be accepting (because we have other odd priorities present)

this is what we did for coBüchi

How do we fix this?

EXPLORABILITY

introduced by Hazard Kupferberg

DEFINITION:

For a natural number $k \geq 1$, nondeterm. parity automaton is **k-explorable** if Eve wins the following k-HD game on it — where she has k tokens

objective :- produce an accepting run on atleast one of her tokens if Adam's word is in the language of the automaton

EXPLORABLE :- If it is K- explorable for some k.

Hazard, Kuperberg 2023

Lemma 7.9: B explorable parity automaton ∧ Eve wins the 2 token game on B

⇓

B is HD

Lemma 7.10: A $[1, K+2]$ automaton — 1 safe coverage

— Eve wins 2 token game from all coreachable triples

⇓

A is explorable

PUTTING EVERYTHING TOGETHER:

$K \geq 1 \in \mathbb{N}$, $\forall$ $[0,K]$ Automaton A, Eve wins 2-token game on A iff A is HD

$\Downarrow$

$\forall$ $[1,K+1]$ automaton A, Eve wins 2 token game iff HD

Proof. $A \rightarrow$ simulation equivalent subautomaton B *

$B \downarrow$ *

2 priority reduction $\leftarrow$

Eve wins 2 token game from coreachable states — *

$B \downarrow$ non-safe coverage $\rightarrow$ $B \downarrow$ explorable $\rightarrow$ $B \downarrow$ HD

A HD $\leftarrow$ simulation equivalent B HD

Final Remarks

Deciding HD for Parity automata

→ Fixed Parity ⟶ P

→ Parity as a Part of Input → PSPACE → upper bound

(also applies for alternating parity automata)

open: what is the exact complexity of Deciding HD for nondeterministic Parity automata whose Parity Index is not Fixed